

Complex Bordism and Cobordism Applications

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Main goals:

- To describe the main notions and constructions of bordism and cobordism;
- To formulate the basic results that are in the heart of these theories;
- To introduce the invariants of stable homotopy groups of spheres in terms of complex cobordism geometry.

Program:

- 1) Vector bundles and their Thom complexes, tangent and normal bundles.
- 2) Structures on closed manifolds, bordisms and cobordisms of such structures, framed, stable complex and 2-complex structures.
- 3) Transversality theorems and Pontryagin-Thom construction, applications to the cobordism rings.
- 4) Spectrum; homology and cohomology theories defined by spectra.
- 5) Thom spectra for framed bordism, complex bordism and 2-complex bordism, Milnor-Novikov theorem and applications to the ring of 2-complex bordism.
- 6) Geometrical construction of invariants of stable homotopy groups of spheres defined by complex and 2-complex bordism theories.
- 7) A survey of calculation methods for invariants. Results and problems.

Suggested topics:

- 1) Landweber-Novikov algebra, Novikov theorem on the structure of all cohomology operations in complex cobordism.
- 2) Adams-Novikov filtration for the spectrum of spheres in terms of Thom spectra and Adams-Novikov spectral sequence.
- 3) Characteristic numbers and the problem of representatives in complex and 2-complex bordism classes.
- 4) Structures on manifolds with boundary and corners, bordisms of such structures, Conner-Floyd theorem and its generalization for n -complex bordism.
- 5) Buchstaber filtration for the ring being dual to the Landweber-Novikov algebra as Hopf algebra; characteristic numbers of manifolds with corners. Application to the E_2 term of Adams-Novikov spectral sequence.
- 6) Construction and calculation method for stable homotopy groups invariants using n -complex cobordism theory

Preface:

The mini-course is devoted to the applications of differential topology and stable homotopy theory methods in constructing invariants of stable homotopy groups of spheres. The initial source of this methods turned out to be the L. S. Pontryagin construction and his result stating that homotopy groups of maps of spheres can be identified with groups of equivalence classes (cobordism groups) of smooth manifolds with a fixed trivialization of their normal bundles. R.Thom's work made a strong influence on the development of Pontryagin's approach, in which the notion of the Thom space of a vector bundle was introduced as a key generalization of spheres -- Thom spaces for trivial vector bundles.

Thom generalized the Pontryagin's construction and proved that the homotopy group of maps of spheres into a Thom space of a vector bundle ξ over a smooth manifold B can be identified with the group of equivalence classes of manifolds with a fixed structure in their normal bundles induced by ξ via some mapping of the manifold to the base B of the bundle ξ .

In the heart of our mini-course we put the cobordism ring of manifolds with a complex structure fixed in their normal bundles.

The methods we are going to present are essentially relying on the fundamental J. Milnor-S. P. Novikov theorem which asserts that this ring is a graded polynomial ring $Z[a_2, \dots, a_{2k}, \dots]$ with $\deg a_{2k} = -2k$. We also need the cobordism ring of n -complex manifolds with a structure of decomposition into sum of n -complex bundles in their normal bundles being fixed. This ring is isomorphic to $Z[a_{2,1}, \dots, a_{2,n}, \dots, a_{2k,1}, \dots, a_{2k,n}, \dots]$ with $\deg a_{2k,i} = -2k$.

A notion of spectrum and construction of generalized homology and cohomology theories in terms of stable homotopy groups of spectra are in the basis of the stable homotopy theory.

A stable structure on manifolds is introduced via a sequence of bundles. These sequences give spectra of Thom spaces and the corresponding homology and cohomology theories will be theories of bordisms and cobordisms as they were named by M. Atiyah.

We shall describe the Hopf, Adams, Conner and Floyd, Novikov invariant constructions for stable homotopy classes of maps of spheres by using the theory of n -complex cobordisms, $n=1,2$.

We listed the program of our mini-course. Its content is oriented towards a broad audience. We include the appendices D and E from the monograph by V. M. Buchstraber and T. E. Panov "Toric Topology" in this booklet as appendices A and B. A number of basic notions that are used in the mini-course can be found in these appendices. The notions we need from stable homotopy theory can be found in the J.F.Adams monograph "Stable homotopy and generalized homology".

In 1967 S.P.Novikov published the paper "Methods of algebraic topology from the viewpoint of cobordisms". The main results of this fundamental work are:

--- the algebra of cohomology operations in complex cobordism was described including Hopf subalgebra of operations, which became well-known as Landweber-Novikov algebra;

--- the Adams-Novikov spectral sequence was introduced and its E_2 term described in algebraic terms;

--- the formal group of geometrical cobordisms was introduced and its applications shown in the classical problem of the cyclic group actions on smooth manifolds.

These results of S. P. Novikov determined the directions of the further development of complex cobordism theory and its applications. We include a recent survey of the author that was devoted to these directions in the booklet. The method of formal groups in complex cobordism theory is the focus of this survey. In 1969 D. Quillen published the outstanding result: the formal group of geometrical cobordisms over a cobordism ring of stably complex manifolds can be interpreted as the universal 1-dimensional formal group over the Lazard ring. General algebraic theory of formal groups became a part of the apparatus of complex cobordism theory due to this result. Based on the general formal group theory Quillen obtained an important result already in this work. This result is: the universal description of the Brown-Peterson cohomology theory and construction of the algebra of cohomology operations in this theory. The Quillen's results are essentially used in the works devoted to the problem of calculation of stable homotopy groups of spheres by the Adams-Novikov spectral sequence in the Brown-Peterson theory.

Nowadays the method of formal groups in complex cobordism is very powerful for providing applications of complex cobordism theory in solving the problems including the one of calculation of invariants that we are going to discuss in this mini-course. One of the first surveys published on this topic was V. M. Buchstaber, A. S. Mischenko, S. P. Novikov "Formal groups and their role in the apparatus of algebraic topology" (1971). Application of the theory of formal groups to the problem of stable homotopy groups of spheres calculation with Adams-Novikov spectral sequence in Brown-Peterson theory can be found in the monograph by D. Ravenel "Complex cobordism and stable homotopy groups of spheres". The basic facts about the formal group of geometrical cobordisms and its applications one can find in the author's survey "Complex cobordism and formal groups" mentioned above. The main topics of this survey are:

--- the classical problem of compact group actions on manifolds;

--- the theory of Hirzebruch genera and their applications in the problem of relations between characteristic numbers of manifolds;

--- an approach to the problem of rigidity of equivariant Hirzebruch genera based on the general theory of formal groups and the theory of Abelian functions.

We do not mention the method of formal groups in this mini-course as this course is a short one. Nevertheless, the material we present here will enable one to use this method effectively for the problems discussed in the mini-course.

We included a list of topics that develop the material of our mini-course in the booklet. The key point will be the geometric realization of the filtration $S = X_0 \leftarrow X_1 \leftarrow X_2 \leftarrow \dots$ giving the Adams-Novikov spectral sequence for spheres. In this filtration the X_1 spectrum coincides with the one introduced by P. E. Conner and E. E. Floyd and the factor-spectrum X_0/X_1 is the spectrum of complex cobordism. Conner and Floyd interpreted stable homotopy groups of X_1 as cobordism groups of manifolds with a complex normal bundle in which the trivialization over the boundary is fixed. A remarkable fact is that all the spectra X_n of this filtration and their factor-spectra X_n/X_{n+1} , $n \geq 1$, are given by the Thom spectra of n -complex cobordisms and their homotopy groups can be identified with the groups of equivalence classes of manifolds with corners, in which the normal bundle has a fixed structure of decomposition in a sum of n complex bundles. We deal with manifolds here which boundaries consist of the "properly" glued n pieces and after restricting the normal bundle to each piece of the boundary one gets the fixed trivialization of the corresponding summand in the normal bundle decomposition.

Thus, the additional list of topics suggests the directions for further research. The following topics will be of special interest: to find the interrelation between the theory of n -complex manifolds and manifolds with corners and the toric topology.

The list of recommended textbooks and papers can also be found in this booklet.

Acknowledgements

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Textbooks

- 1) J. F. Adams, "Stable homotopy and generalized homology", Univ. Of Chicago Press, 1974.
- 2) V. M. Buchstaber and T. E. Panov, "Toric Topology", AMS Mathematical Surveys and Monographs, Vol. 204, 2015.
- 3) P. E. Conner and E. E. Floyd, "The relation of cobordism to K-theories", Lecture Notes in Mathematics, Vol. 28, 1966
- 4) Morris W. Hirsch, "Differential topology", Graduate Texts in Mathematics, Vol.33, 1976
- 5) D. Ravenel, "Complex cobordism and stable homotopy groups of spheres", second edition, AMS Chelsea Publishing, AMS, Providence, RI, 2004
- 6) R. E. Stong, "Notes on cobordism theory", Princeton Univ. Press, Princeton, NJ, Univ. Of Tokyo Press, Tokyo, 1968.

Recommended papers and surveys

The numbers 1, 7, 8, 9, 11, 12, 17, 22, 25, 36, 38, 41, 54, 63 in the list of references to the survey V. M. Buchstaber, "Complex cobordism and formal groups", Russian Math. Surveys 67:5 (2012), 891-950.

And in addition:

M. F. Atiyah, "Thom complexes", Proc. London Math. Soc., (3) 11, 1961, 291-310.

Gerd Laures, "On cobordism of manifolds with corners", Trans. of the AMS, Vol. 352, No. 12, 2000, 5667-5688